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Targeting unstable periodic orbits

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Specific targeting of unstable periodic orbits has been achieved by using large-amplitude perturbations in a dynamical system. The method has been demonstrated experimentally on a CO_2 laser with modulated losses whose unstable periodic orbits are created either at a period-doubling or at a saddle-node bifurcation. Applications of the technique to switch the phase of the dynamical motion, to set boundaries of basins of attraction, and to measure Floquet multipliers are discussed.

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In the early studies on nonlinear dynamical systems, the interest was centered on attractors since they represent the long-term behavior which is observable both in numerical simulations and real experiments. More recently, it appeared that unstable periodic orbits (UPO's) were an extremely interesting characteristic of such systems. They could be used successfully to control chaotic systems and stabilize them in periodic regimes [1]. Unstable periodic orbits are also the basis of a method of communication using chaos [2]. Moreover, their topological invariants allow one to unfold the complexity of chaotic attractors by determining the template on which they are wound [3].

Until recently, the method to reach these UPO's was to use the ergodicity of the chaotic attractors. The dynamical system explores its phase space until it reaches the vicinity of the desired UPO, then tiny corrections allow one to keep the system in this particular periodic state. Recently, Shinbrot *et al.* proposed an iterative method to reduce the time required to reach a special region of the phase space using the exponential sensitivity of a chaotic system to small perturbations [4]. In this paper we use a single-step trial and error method to reach UPO's. We show that suitable timing of large perturbations allows one to target specifically such UPO's and possibly to set the boundaries of basins of attraction. Contrary to the control methods, the technique developed here uses large-amplitude perturbations. We show experimentally that strong corrections may keep the high selectivity required to set the system in the close vicinity of the unstable cycle.

The method has been implemented on a CO_2 laser with modulated losses, a system which has been demonstrated to evolve towards chaos via a period-doubling cascade and simultaneously displays the sequential formation of horseshoes [5]. Therefore the CO_2 laser with modulated losses allows one to illustrate targeting in different situations since the targeted UPO may be a periodic orbit which destabilized in a period-doubling bifurcation or an unstable orbit created in a saddle-node bifurcation responsible for the creation of coexisting attractors [5].

In the first case, the target UPO is embedded inside the basin of attraction of the initial state and fluctuations will always make it evolve back to the initial state. In the second case the UPO which is targeted was created in a saddle-node bifurcation. Therefore it sets the limit of two basins of attraction and the system will evolve through fluctuations towards either of the two attractors whose basin of attraction is limited by the UPO.

In our case the method has been applied to a nonautonomous (driven) system which makes the timing procedure easier. Nonetheless, it is of general use and is also valid for autonomous systems. The purpose of this paper is to show that the trial and error approach is efficient because there exists generically a solution to the problem of one-step targeting of UPO's. For instance, in a three-dimensional (3D) phase space, the UPO and its stable manifold define a 2D manifold. Starting from an initial point on the attractor, a perturbation with suitable strength and sign will intersect this 2D manifold and lead to a trajectory evolving towards the UPO. For a perturbation which does not reach exactly that 2D subspace, the divergence from the UPO tells us how close to the UPO the system was sent by the targeting pulse and allows us, by the same algorithms as those used for the control of chaos, to reach the target. The experiments reported here take explicit advantage of the access to two control parameters, namely, the amplitude and the phase of short perturbations. In the general case, the amplitude and the timing (here the phase) of the targeting pulse allows one to span a 2D subset of the phase space which generically intersects the UPO's and has a 1D set of intersections with their stable manifold in the case of the 3D phase space. Therefore for each timing there should be one amplitude leading to the targeted subspace. The other parameter allows one to tune the perturbation in order to reach the UPO. Let us emphasize once again that this is achieved by a trial and error learning procedure which compares the results of two targeting pulses reaching the vicinity of the desired target.

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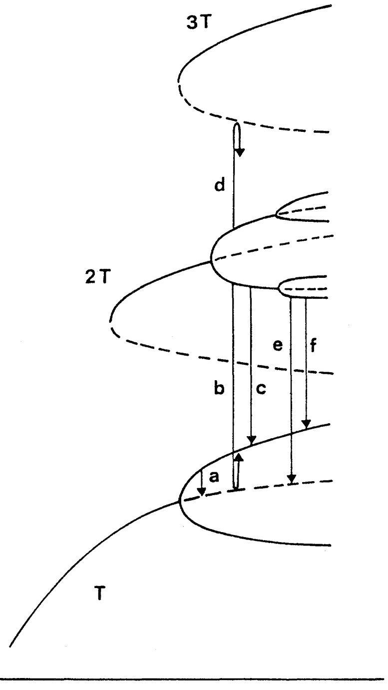

FIG. 1. Bifurcation diagram of the CO_2 laser with modulated losses versus amplitude of the modulation. Full (dotted) lines correspond to stable (unstable) cycles. The arrows illustrate the effect of the pulsed loss perturbation and letters refer to the recordings of Fig. 2.

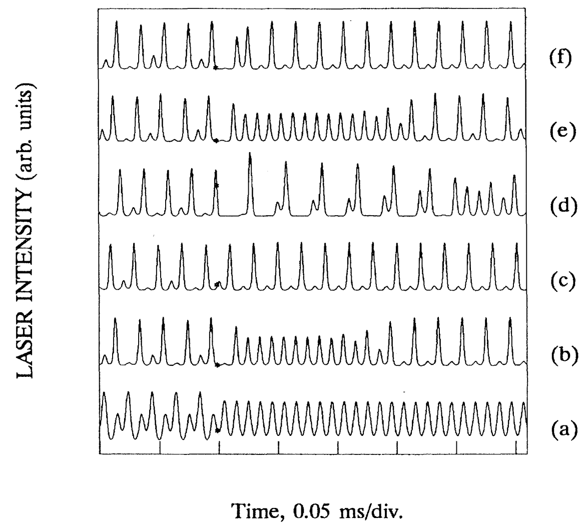

FIG. 2. Examples of switching from stable orbits to unstable ones for different initial states and pulse perturbation: (a) $2T_s \rightarrow T_u$, (b) $4T_s \rightarrow T_u$, (c) $4T_s \rightarrow 2T_s$, (d) $4T_s \rightarrow 3T_u$, (e) $8T_s \rightarrow T_u$, (f) $8T_s \rightarrow 2T_s$. The notation $nT_s \rightarrow mT_u$ indicates that the initial dynamical state is the nT stable orbit and the final state, i.e., the target, is the mT unstable orbit. The time of the pulse perturbation is shown by dots: here the modulation frequency (100 kHz) is twice the laser relaxation frequency.

Experiments have been performed on a single mode CO_2 laser with an intracavity acousto-optic modulator operated at 100 kHz. Short lived perturbations are caused by optically induced ir absorption of nonequilibrium charge carriers (NCC's) in a GaAs window of the CO_2 laser tube. NCC's are excited in the impurity band by illuminating the laser window with 15 ns pulses from a Q-switched neodymium-doped yttrium aluminum garnet (Nd:YAG) laser, as described in [6]. The rise and fall times of the losses have been estimated by the method proposed in [7] as 50 and 300 ns, respectively. They are significantly lower than the modulation period of 10 μs, therefore allowing for a pulse description of the perturbation. The output intensity is monitored by a $Hg_xCd_{1-x}Te$ photodetector and a digital oscilloscope at a time resolution of 50 ns and further computer processed. Several clock and delay modules allow one to synchronize and choose the phase of the pulsed perturbation for any periodic nT regime with $n \leqslant 8$.

This laser is well modeled by equations for the population inversion D and the laser intensity I

$$\dot{I} = 2\kappa I[AD-1],$$

$$\dot{D} = \gamma[1-D-DI],$$

where κ is the cavity damping rate, γ the population inversion rate, and A the pump parameter. In our experiments cavity losses are sinusoidally modulated at a frequency ω_n close to the relaxation frequency of the system. Moreover, the targeting pulse is also applied to this parameter. Therefore $\kappa = \kappa_0(1+m \sin\omega_n t) + \delta\kappa(t)$, where $\delta\kappa(t)$ describes the targeting pulse. Note that this is specific to our device. In the fiber laser, for instance, the targeting pulses could more easily be applied to the pump parameter A. The bifurcation diagram of the laser used in the experiments is shown in Fig. 1 and the different targets which were reached are indicated by the arrows.

Let us first consider the case where the targeted UPO has been created in a period-doubling bifurcation, i.e., it is embedded inside the basin of attraction of the initial orbit. As mentioned earlier, the unstable manifold of the UPO is connected to the stable manifold of the initial state and, whatever the perturbation, the system will evolve back to the initial state, unless it reaches exactly the stable manifold of the UPO. The phase of the perturbation, i.e., its timing with respect to the modulation in a driven system or the stable oscillation in an autonomous system, sets the direction of the perturbation induced shift in the Poincaré section and in particular which of the two intersections will be the initial state. Therefore for any phase, at least within a half-period because of sign problems, there exists a perturbation amplitude which allows the system to reach the target. An example of such targeting is given in Fig. 2(a), where the laser is sent onto the T orbit which destabilized in the period-doubling bifurcation. A series of similar experiments carried out for different amplitudes of the periodic external driving allows one to follow the evolution of a particular unstable periodic orbit. Figure 3 gives an example of such an investigation in the case of the T-unstable orbit for whose amplitudes the maxima and the Floquet multipliers as measured from the signal divergence have been reported. The T-unstable orbit obviously has am-

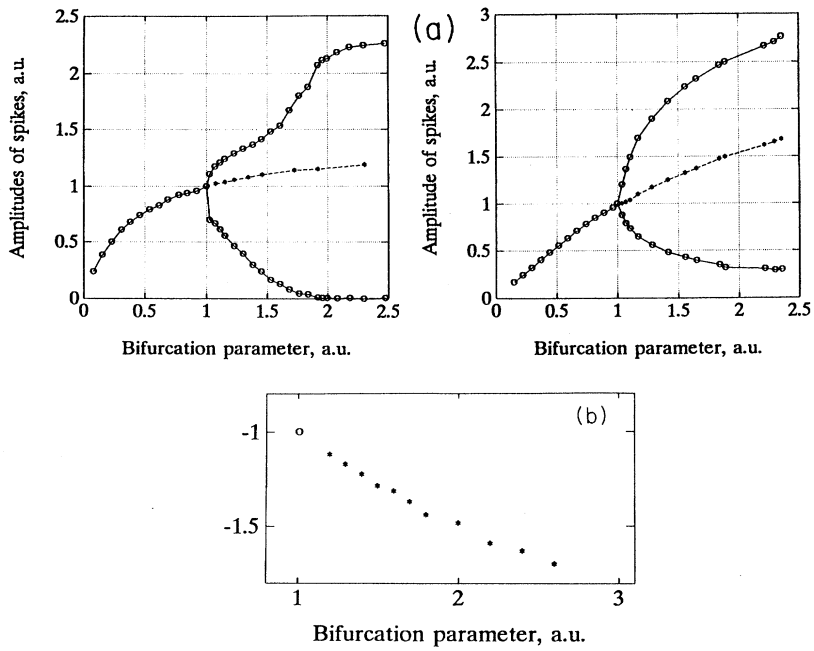

FIG. 3. (a) Part of the bifurcation diagram showing the laser intensity maxima versus modulation amplitude near the T-$2T$ bifurcation. Amplitudes of the maxima corresponding to the T_u orbit are shown to be in continuity with those of the T orbit before the period-doubling bifurcation. (b) Dependence of the Floquet multiplier λ_a on the bifurcation parameter (in units of the threshold for the T-$2T$ bifurcation).

plitudes in the continuation of the T orbit from which it originates and as expected its Floquet multiplier tends to -1 at the bifurcation point. Targeting optimization is achieved in two steps. First, for a given phase, the amplitude of the pulse perturbation is chosen so as to reach exactly the stable manifold of the UPO, then the amplitude is reduced and the phase continuously matched as long as the targeting condition can be met.

The relationship between the amplitude and the moment (phase) of switching the pulsed losses is a characteristic of the targeting since for each phase there exists a specific amplitude that meets the targeting conditions. The evolution of this amplitude versus the phase is plotted in Fig. 4 together with the $2T$ response of the driven laser which is the initial dynamical state. It shows the existence of the optimum tar-

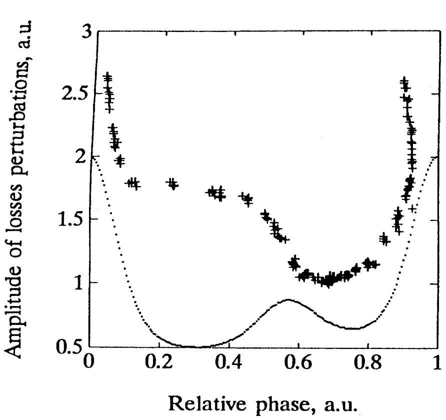

FIG. 4. Dependence of the amplitude of loss perturbations matching the targeting condition on the phase in 2π units for the $2T$ cycle. Dots show the $2T$ stable cycle for reference. The bifurcation parameter is 1.25 (in units of the threshold for the T-$2T$ bifurcation).

geting region. In the case of the driven CO_2 laser with a bifurcation parameter of 1.25 (in units of the first threshold for period doubling), the optimum is obtained for a phase of about 0.7, which remains about the same as the bifurcation parameter is increased up to the second ($2T$-$4T$) bifurcation while the (threshold) amplitude values increase in accordance with the increased separation between the unstable T and stable $2T$ orbits. Note also that in the phase region between 0.8 and 0.9 there are amplitudes of the loss perturbation that allow the laser to reach the T-UPO.

The particular situation of the T-UPO created in T-doubling bifurcations also allows us to use the perturbation technique to flip the phase of the laser output by approximately π. The $2T$ UPO is embedded inside the basin of attraction of the T orbit, in the vicinity of this orbit. Therefore the two orbits coexist in neighboring regions of the phase space but neighboring points of the T correspond to a phase difference of π. For a given phase, if the perturbation is small, the system returns in the vicinity of the initial state. There exists a given amplitude for each phase that allows the system to reach the UPO and, if the pulse amplitude is too large, the system precipitates to a stable periodic orbit but with a phase shift of approximately π. Note that as the system reaches exactly the UPO, it eventually breaks the symmetry and evolves towards one phase state or the other depending on noise and fluctuations. A significantly more complicated dependence between the amplitude of loss perturbations and the relative phase exists in the case of the switching from the stable $4T$ regime to unstable T and $2T$ orbits. In particular, at the same amplitude of pulse losses, we can switch to T and $2T$ depending on the relative phases. The threshold value for pulse losses discussed above is obviously different for the two orbits and is smaller for the $2T$ orbit. This coincides with common intuition at least near the bifurcation point since the $4T$ orbit originated from the doubling of the $2T$ cycle.

The situation is different when two attractors coexist. Then, for topological reasons, the boundary of their basins of attraction is an unstable region, usually an unstable cycle which can be reached from either attractor. The relative positions of the different dynamical regimes of the modulated CO_2 laser are illustrated in Fig. 1. For large enough modulation amplitudes, the laser presents generalized bistability between the original attractor or its period-doubled variations noted as 2^nT (n integer) and the stable cycles 2^nmT (m integer) originating from the sequential horseshoe formation. The recordings of Figs. 2(b), 2(c), and 2(d) show that, starting from the same initial dynamical state, here a stable $4T$ orbit, different states may be aimed at, depending on the phases of the pulse perturbation. In the recordings of Figs. 2(b) and 2(c), the 2^nT attractor is aimed at, and depending on the phase and/or amplitude, it is possible to steer the system to the $2T$ stable orbit [Fig. 2(c)] or to the T unstable orbit [Fig. 2(b)] if the perturbation is optimized. For a completely different phase, the pulse perturbation pushes the laser to the $3T$ unstable orbit [Fig. 2(d)]. Figures 2(e) and 2(f) present a similar situation with the $8T$ stable cycle as the initial regime and the unstable T or the stable $2T$ orbits as final states, according t[illegible] perturbation.

These effects are related to the noise-induced hopping between different attractors studied by Arecchi *et al.* [8] and the elimination of multiple basins of attraction using chaos [9] except that here we have used a single-shot pulse instead of noise or chaotic driving, i.e., the perturbation is applied only for a very short time instead of continuously applied perturbations as in the previous works. Note that here we reach the UPO's of the unperturbed system while, by using continuous perturbations, the nature of the system is altered. Until recently, unperturbed UPO's were accessible only in chaotic systems because in that case they are embedded inside the chaotic attractor. They could be observed in parameter domains where the system is not chaotic by using a tracking technique in which the system is stabilized in these orbits while chaotic and then brought to the nonchaotic region [10]. The method proposed here allows the system to reach directly these unstable orbits and uses the phase sensitivity of the system to minimize the required perturbation. We have shown in the example of the CO_2 laser with modulated losses that phase switching and measurement of Floquet multipliers are easily achieved by using this targeting technique.

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